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Incorporating Inquiry-Based Learning in the Calculus Sequence:

A Most Challenging Endeavour.

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M. Padraig M. M. McLoughlin, Ph.D.
265 Lytle Hall,
Department of Mathematics,
Kutztown University of Pennsylvania
Kutztown, Pennsylvania 19530
mcloughl@kutztown.edu

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Abstract

Incorporating Inquiry-Based Learning in the Calculus Sequence: A Most Challenging Endeavour.

M. Padraig M. M. McLoughlin
Department of Mathematics,
Kutztown University of Pennsylvania

A course in the Calculus sequence is arguably the most difficult course in which inquiry-based learning (IBL) can be achieved with any degree of success within the curriculum in part due to 1) the plethora of majors taking Calculus to which the sequence relates to their majors in what is considered an 'applied' manner; 2) the sequence is intertwined such that 'coverage' matters since if a critical concept or area was not 'covered' in Calculus I or II it might do serious harm to the student in Calculus II, III, or beyond where the understanding the topic may depend significantly on said material which was not 'covered.'

So, this paper argues (pedagogical and practical justification are submitted) for use of a modified Moore method (MMM) which employs elements of the classic Moore method (students doing rather than seeing, hearing, or reading) which creates a moderate pace for the course; not too fast (as perhaps in a traditional German seminar (recitation) method) nor too slow (as perhaps in a constructivist or pure Moore method course) and presents the model for use the MMM in the Calculus sequence. Further, it is proposed that the MMM assists students to establish a firm foundation for subsequent course work and creates an excellent potential for students to have the possibility to master the material. The author of this paper has experienced teaching such courses in the Calculus sequence in such a manner for approximately twenty-five years.

I. Introduction, Background, and a Brief Overview of the Moore Method

One can argue, probably quite convincingly, that a course in the Calculus sequence[1] is arguably one of if not a most difficult course in which inquiry-based learning (IBL) can be achieved with any degree of success within the mathematics canon. For it is the case that a plethora of majors take at least one of the courses in the Calculus sequence to which the course and material relates to their majors in what is considered an 'applied' manner; further, the Calculus course sequence is intertwined such that 'coverage' matters since if 1) a critical concept or area was not 'covered' in Calculus I it might do serious harm to a student in Calculus II, Calculus III, or beyond, 2) a critical concept or area was not 'covered' in Calculus II it might do serious harm to a student in Calculus III or beyond, 3) a critical concept or area was not 'covered' in Calculus III it might do serious harm to a student beyond the sequence. Such is the case since the understanding of a topic may depend *significantly* on said material which was not 'covered.'[2]

However, if an instructor or instructors teaches a select group of students and IBL is the manner of instruction throughout the Calculus sequence[3] taught then this discussion is of little to no use. Furthermore, we assume that the course is taught at an 'average' not 'elite' college or university and is of the depth and breathe defined in [5] or [6] (Committee on the Undergraduate Program in Mathematics, (CUPM)) 1963 and 1965 guides) rather than at an advanced placement (AP) high school level. Moreover, the frame-work of this paper assumes that IBL is not the manner of instruction throughout the Calculus sequence and there is not a sequestering of mathematics majors from the general population taking Calculus at the university (as is the case at Kutztown University of Pennsylvania (KUP) where no such sequestering takes place). Also, before proceeding, it should be noted there are some fine materials available for the Calculus sequence that are IBL from a frame-work that is a Moore or modified Moore method from the Legacy of R. L. Moore Project (http://www.discovery.utexas.edu/rlm) which is funded in part by

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[1]The Calculus sequence is understood to mean in a semester system a body of courses of which the subject is defined in the Committee on the Undergraduate Program in Mathematics, Pregraduate Preparation of Research Mathematicians, Washington, DC, Mathematical Association of America, 1963; and, Committee on the Undergraduate Program in Mathematics, A General Curriculum in Mathematics for College, Washington, DC, Mathematical Association of America, 1965. It is typically understood to be three four-credit semester courses named Calculus I – II – III. There are variations on this: for example, at Louisiana State University there exists Calculus I – II – III of 5 semester credits, then 4 semester credits, then 3 semester credits; at Kutztown University of Pennsylvania there exists Calculus I – II – III - IV of 3 semester credits each; and, at schools where the quarter system is employed there is a Calculus I – II – III or Calculus I – II – III – IV whose quarter hours convert to approximately 12 semester hours.

[2]It seems to be the case that with many students the background High School material is not recalled, was never learnt, or was not 'covered' in a manner which was preparatory for college-level mathematics. Such *seems* to be *more* the case now than twenty years ago and *seems*, at least in part, due to an 'integrated' curriculum in High School and the use of calculators. Anecdotally, few, if any students in some classes can explain what $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x^2$ is, what its graph is in the plane, etc. There seemed to be more students who knew what such was twenty years ago than now. Such is the case with High School or before –level arithmetic, algebra, geometry, and basic or transcendental functions.

[3]As is the case, it seems, at the University of Chicago. See Herrmann, Diane (1046-97-730) this conference (2009), for example.

the Educational Advancement Foundation in Austin, Texas (for example: *Calculus II* by W. T. Ingram, Robert Roe, and Leon Hall at the University of Missouri – Rolla used in conjunction with the Stewart *Calculus* book I believe and the note of the esteemed Professor of Mathematics at Emory University, W. S. Mahavier, *A Moore Method Calculus II Course*).

We also assume that mathematics is a subject that is broad (not unary) and that Calculus is but one branch of mathematics; but, it is a 'gateway' topic to most of the higher and more abstract areas of mathematics. Also, the canon for university Calculus is stable and might shift slightly but there is an overall established, basic standard, and agreed upon by most mathematicians body of mathematics knowledge that different students from different places can understand in the same or similar *correct* ways but there might be other ways that may also be correct that derive the correct solution to a problem in standard mathematics that are not known or are not standard and might be developed.

Returning to a discussion of the Calculus sequence, if a student does not understand a derivative, how would that student truly understand an indefinite Riemann integral? Suppose a student does not understand a univariate definite Riemann integral; how then does that student understand double or triple definite Riemann integrals? How does a student understand differential equations without understanding Calculus? How does a student understand Real Analysis, Complex Analysis, Numerical Analysis, Probability Theory, Difference Equations, Dynamical Systems, etc. without understanding Calculus? Central to these questions is not only Calculus, but what does it mean for a person to understand a topic or subject?

Let us review the idea of the question of, "what does it mean for a person to understand a topic or subject?"[4] Many would *claim* to understand much because they are familiar with something, have heard of it, read of it, or been told of it. Others might claim to understand much because they are told of it by an 'authority,' read it in text on the subject, or because they habitually believe it. But that is *not* what we will mean when we say to understand something. Let us agree we mean that person A understands[5] thing B if and only if he is 1) able to comprehend it; to apprehend the meaning of or import of, 2) to be expert with or at by practice, 3) to apprehend clearly the character or nature of a thing, 4) to have knowledge of to know or to learn by information received, 5) to be capable of judging with knowledge, or, 6) the faculty of comprehending or reasoning. Such a definition complements Bloom's taxonomy[6] and focuses the discussion on the idea of thinking. A person can only comprehend when that person is thinking so thinking is an antecedent to understanding - - no thought, no understanding. So, mathematics education should be centred on encouraging a student to **think** for one's self: to conjecture, to analyse, to argue, to critique, to prove or disprove, and to *know* when problem is solved correctly, to *know* when an argument is valid or invalid.

Perhaps the unique component of mathematics which sets it apart from other

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[4]See McLoughlin, Inquiry-Based Learning: An Educational Reform Based Upon Content-Centred Teaching (1046-97-644), this conference (2009). This is almost a compleat repetition of a part of that paper but is worth including in this paper for clarity and that it might be the case that one might read this paper and not that paper.

[5]Oxford Universal Dictionary (1944), 3rd *edition*, Oxford University Press: London, UK.

[6]See Sax, Principles of Educational and Psychological Measurement and Evaluation (1989) page 72.

disciplines in the academy is the demand for succinct argument from a logical foundation for the veracity of a claim. The author of this paper submits that in order for students to learn, students must be *active* in learning. Thus, the student must learn to understand a problem and solve it precisely, accurately, and correctly (not just 'get' an answer by 'any means'). The student must learn to conjecture and prove or disprove said conjecture.

One cannot learn to solve problems by reading a book, we learn to solve problems by problem-solving.[7] One cannot learn to conjecture from a book, we learn to conjecture by conjecturing![8]

Ergo, the author of this paper submits the thesis that learning requires doing; only through inquiry is learning achieved; and, hence this paper proposes a philosophy such that the experience of creating an idea and a mathematical argument to support or deny the idea is a core reason for an exercise and should be advanced above the goal of generating a polished result. Indeed, when stating that students must be *active* in learning and that inquiry-based learning (IBL) enacted via a modified Moore method (MMM) (see [34] through [41] for detailed background of the authors' modified Moore method and its use in courses other than Calculus) is an authentic way to actualise a learning environment where the content studied is the centre of the experience, and that IBL is a content-driven pedagogy; as such it is content-centred not instructor-centred or student-centred[9] it is meant in compleat and utter contradiction to what appears to be the 'established educational' understanding of said. There seems to be an agreed upon distinction made between instructor-centred and student-centred learning in the literature but there is also some difference in educational research presented as to content-centred and student-centred learning; e.g.: "active learning is a buzz phrase that captures the teaching technique promoted by learner-centered [sic] as opposed to content-centered [sic] instruction."[10] Moreover, even the idea of *active* learning does not to be an entirely clear concept.

So, the author proposes that two particularly important components of inquiry-based learning (IBL) through the MMM is that it creates an ideal setting for later undergraduate research and that it engages the student in better grasping material from previous courses because the student must *use* pre-requisite material in a Calculus class taught in a modified Moore method (MMM) manner. So through inquiry the student is encouraged to probe deeper into a subject that not only includes the one under discussion but those before.

In a Calculus class taught via this modified Moore method (MMM) much focus is on material from previous courses because the student must justify and explain why a solution is correct, how the student derived the solution, when the student

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[7]This statement is not meant to be sarcastic but to demonstrate that there is idempotency within the meaning of the words.

[8]Ibid.

[9]See McLoughlin, Inquiry-Based Learning: An Educational Reform Based Upon Content-Centred Teaching (1046-97-644), this conference (2009).

[10]See Halonen, Brown-Anderson, & McKeachie, "Teaching Thinking," in *McKeachie's Teaching Tips: Strategies, Research, and Theory for College and University Teachers* (11th ed.), Boston: Houghton-Mifflin, 2002 and Yoder, & Hochevar, "Encouraging Active Learning Can Improve Students' Performance on Examinations," Teaching of Psychology 32, no. 2 (2005) page 91.

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shows whether or not a claim is valid, or even when the a student makes a claim!

In the MMM Calculus class when a student makes a claim oft the author asks the claimant why he is claiming the principle, idea, or methodology. It is hoped that this query aides the student in understanding the (perhaps subtle) difference between and betwixt believing, opining, and knowing and help the claimant, the other members of the class, and the author attempt to comprehend where the idea arises from. Often after the, "why," is asked the claimant argues the veracity or lack thereof of the claim on the spot! There are times when other students in the class will argue for or against the idea (which is sometimes good but other times 'irritating' for the author wishes for the idea to be 'chewed over' perhaps for a time) with the claimant in class. Such discussion is facilitated and directed by the author whilst the author attempts to interject little of his ideas unless the discussion veers too far from the topic or is reinforcing incorrect notions. The author opines that by engaging in said experience, students can decipher whether they have is a true interest in the field or not.

II. The Challenge of Inquiry-Based Learning Using a Modified Moore Method (MMM) in A Calculus Course

A superficial understanding of many subjects is an anathema to a Moore adherent; a Moore adherent craves a deep, full, and compleat (as compleat as possible) understanding of a subject (or subjects).[11] 'Coverage' of material is not a hallmark of the Moore method. On the other hand, traditional methodology includes the pace of the class set by the instructor (usually prior to the semestre). 'Coverage' of material is a trademark of traditional methods. Maximal treatment of material is typical in a traditional classroom. So, what do we do about balancing the depth and breathe, especially in the context of the course being taught at an 'average' university with a heterogeneous population of students?

In the IBL Calculus class as described herein neither the instructor nor student creates the syllabus[12] – it is imposed from outside the classroom (as is the case at most colleges or universities; it is a standard departmental syllabus) – but the pace is dictated to a degree by the students and is regulated and adjusted by the instructor and what material is accented for a more deep analysis is decided upon by the instructor. Hence, the author's MMM shares a commonality with traditional methods in so far as pacing is concerned; we acknowledge that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be not answerable at the moment. Therefore, the author's MMM seeks to balance the question of 'how to' with the question of 'why.' A subject that is founded upon axioms and is developed from those axioms concurrently can be addressed with the questions 'why' and 'how to.'

It is not a problem that the syllabus is departmental because what material is accented for a more deep analysis is decided upon by the author. So, in an IBL Calculus class taught by the author it is in this material for deeper, fuller analysis where the opportunity for the student to engage in authentic learning exists. In the next section of this paper we will concentrate on the material that the author has found is where in a Calculus sequence IBL can truly be actualised.

It is the position of the author that we accept the concept of minimal competency, that a student needs some skills before attempting more complex material, that is to say that there is a set of objectives that the instructor attempts to meet when teaching a class, that he is duty-bound to include that material. The idea of the minimal competency is what the author assumes when a student enters his class – that the student is competent in Arithmetic, Algebra, Geometry, Graphing, and Trigonometry before Calculus I; all that and Calculus I before Calculus II; and likewise along with Calculus II for Calculus III.

The goal of education is not, under the MMM methodology, 'vertical' knowledge (knowing one subject extremely well) nor 'horizontal' knowledge (knowing many subjects superfluously), but this pedagogy attempts to strike a balance between

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[11]See Davis, page 70; Fitzpatrick, "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 44; Fitzpatrick, Some Aspects of the Work and Influence of R. L. Moore, *A Handbook of the History of Topology* 1996), page 9; Forbes, page 194; Paul R. Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985), page 262; and, Edwin E. Moise, "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, 4 (1965): page 409.

[12]See http://www.math.kutztown.edu for standard syllabi or
http://faculty.kutztown.edu/mcloughl for specific syllabi for courses the instructor teaches.

the two. Students are permitted to work together on any non-graded assignments; but, are encouraged not to do so for so long as a student knows a compleat idea rather than parts (as is the case with some group assignments where each person only does a part and then it is compiled or where one person does all the work for a group as is oft the case (especially pre-college)) more than one student working on an assignment is not a concern. However, it seems to be the case that for many students working in groups where each person truly understands each part of each problem is a rarity. Under IBL using our modified Moore method, traditional regularly administered quizzes, tests, and a comprehensive final[13] are a part of a course. However, a part of each quiz or test (no less than ten percent nor more than thirty percent) is assigned as 'take-home' so that the student may autonomously compleat the 'take-home' portion with notes, ancillary materials, etc. whilst the rest form the 'in-class' portion of the assessment. An honour code is a part of each course the author teaches such that all graded assignments must be done by an individual and the individual confers with no one but the instructor.[14]

Ideally the true nature of work is with time allowance; but also includes time constraints. Hence, an artificial time constraint of a class period is imposed upon the instructor and students because of the system in which they work. However, not all meaningful and educationally enriching exercise can be included on a test in a class period; hence, the inclusion of take-home assessments. However, not all assessments should be take home since there should be some measure of retention of key concepts by the student and if all were take home (or open book in class assessment) then the exercise is perhaps more about *finding* information than *understanding and retaining* it. On the other hand, if all assignments were in-class then one could argue that the exercises can and oft deteriorate into students regurgitating trite tid-bits and small parts of concepts rather than engaging in deeper analysis. Nonetheless, practical considerations force the author to note that if all assignments were take-home; there is a contingent of students who cheat – there is no way around this sad fact of modern society and the reality that ethics are in flux; so the author has found he is duty-bound to attempt to make the educational experience fair (or as fair as is within his control and as fair as is humanly possible).

The MMM Calculus class includes class discussion and allows for the discussion to flow from the students but be directed by the instructor. It is should be expected that about one-fourth of many class periods are dedicated discussion of ideas about the definitions, axioms, or arguments. Approximately one-half or more of many class periods involves students presenting their work. The work presented includes solutions to problems from the book, solutions to problems from instructor created worksheets (downloaded from his web-site), solutions to problems from copies of problems from an out-of-print book (photocopied for the students), or claims made

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[13]A comprehensive final is a critical part of the author's methods for it allows the student to take time to reflect on that which was learnt well, learnt, or not learnt and demonstrate a breathe of aptitude with the content rather than a depth as the presentations, quizzes, or even tests.

[14]The student signs a pledge that includes: "*No help from any person other than yourself and from any notes other than your own. However, you may use other books from the library. You may discuss this paper only with the instructor before handing it in to be graded. If you do not understand there directions see the grading policies under cheating. No calculators, computers, etc.*

'I understand the definition of cheating and I received no help from another person nor did I confer with any other person:' <signature of student> "

by students in which students have volunteered to solve. In the IBL Calculus class using our MMM, it is the case that the method allows for applications (minimal discussion of applications exists in the pure mathematics courses since the emphasis is on the foundations of theoretical mathematics) and modelling (with regard to the fact that students present their arguments before the class and that there exist exemplars for the students as well as the students in the class reviewing a presentation critically). The IBL Calculus class using our MMM does *not* include group assignments of any kind.

One other point about the philosophical or methodological underpinnings of inquiry-based learning (IBL) using our modified Moore method (MMM) method bears mentioning: that of personal responsibility. The students are adults and are treated as such. They are not talked down to and are treated as members of a community of scholars. Students are addressed as, "Ms. Surname" or "Mr. Surname," so that the atmosphere created is one that is professional. That the instructor has more experience is true, but that does not imply that the ideas expressed by the students have any less merit than the instructor. There are too many examples of students having ideas that were better than the author's, students who viewed a problem in a more refined manner than the author, or realised solutions to problems that the author had not worked out yet. Since the students are adults, they are held responsible to compleat work in a timely manner; but, if they do not have work completed then they are held accountable. The instructor does not do the work for the student; he leaves them to do their work.

At least one quiz is administered every three to six class periods, part in class part take home, or all take home in which the students are asked to do elementary problems (usually in class), attempt to solve more challenging problems, and outline an argument to prove or disprove conjectures.[15] They are required (of course) to work alone. The quizzes are graded and commentary included so that feedback is more than just a grade. Also, there are three or four major tests during the semester and a comprehensive final; thus, a Calculus course taught in an IBL manner as outlined herein is grading intensive for the instructor. The frequency of the quizzes creates a standard for the students so they do not fall behind. The final and the tests (no less than three tests nor more than four, depending on the length of the semestre) gives the students the ability to demonstrate competency or proficiency over a part of the course and an opportunity and responsibility to digest and synthesise the material which, it is opined, leads to understanding of the material.

The testing schedule differs from a pure Moore method or most current versions of reform methods and shares a commonality with traditional methods. It may be a tad more 'quiz intensive' and time consuming than traditional methods, but the author has found that many of his colleagues who employ traditional methods like grading homework (which is not a part of the author's method) is also rather time consuming so it might be similar to the traditional methods in that regard.

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[15]Rigorous proofs are not expected of student in the Calculus sequence. The author uses the terminology, "show claim A is true or not true," to mean less than a proof but more than some derivation of something from assuming the conclusion as is the case in many books published currently. The author and colleagues teach students methods of proof in the Foundations of Mathematics course at Kutztown University of Pennsylvania. It is the case that most in post-Calculus (any course with Foundations of Mathematics as a pre-requisite) courses students are required to prove or disprove claims.

Experience with many different course sizes over the past twenty years has led the author to conclude that optimal course size is between approximately fifteen and twenty-five. When there are less than about fifteen students, then many a class discussion oft suffers for a lack of interaction. When the class size is more than about twenty-five students, then class discussions are often difficult to facilitate and can be problematic because so many students wish to be heard simultaneously. Also, if the class size exceeds approximately twenty-five, then the burden of grading so many papers becomes quite heavy and the turn around time lengthens which is detrimental. It seems that it is best to provide feedback in a timely manner so that the students have time to reflect on their work and discuss the work in a follow-up class session or during office hours. If too much time has elapsed between the times students hand the papers in and they get the papers back, their memory of *why* they thought what they thought dwindles and the educational experience for the student suffers.

Many policies of the department supersede that of the author. For example, in material discussed, class hours, text, etc. the instructor uses that which has been agreed to by a majority vote of the faculty. However, we opine that in order to teach in an IBL manner the text is supplemented.[16] Yet, it is the case that a majority of students the author has taught have either not been in his class for a pre-requisite course or in a course where the current course is pre-requisite (obviously since the author teaches three or four courses a semester and there are over well over 200 sections of mathematics classes at Kutztown University of Pennsylvania (KUP)).

The author has studied under professors who have taught in many different ways during his formal university educational experience which spans from the late 1970s to the 1990s.[17] Some of his professor include: M. F. Neff, D. Doyle, S. Batterson, and H. Sharpe of Emory University; M. Smith, B. Fitzpatrick, W. Kuperberg, P. Zenor, and C. Reed of Auburn University; and, J. Walker, Y. Hsu, C. Oshima, and J. Neel of Georgia State University. These professors taught classes or directed the research of the author using the Moore, traditional, or reform methods. Hence, the author developed this modified Moore method (MMM) over the years of his college-level teaching experience (1982 – present). It is constantly being analysed, refined, and evaluated so per se it is more dynamic rather than static a system. As such it was similarly created via an action research model [25] in an empirical manner rather than in a quantitative manner.

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[16]I used to supplement in a few courses but since 2000 or so I create supplementary material in every course. I opine this is because many texts which I found acceptable are now out of print and most of the texts I am aware of or are available are full of calculator, computer, or busy work. I could be in error about this but it seems (anecdotally) to be that many texts are becoming 'dumbed-down.' Also, a quote by my esteemed colleague, Dr. Chuang Peng, still rings in my ears, "there is no perfect book," which prompted me to begin formally supplementing books in the early 1990s.

[17]See http://faculty.kutztown.edu/mcloughl/curriculumvitae.html for a compleat curriculum vitae.

III. Organisation of a Calculus Course When Using a Modified Moore Method (MMM)

In each Calculus course the author teaches,[18] the first meeting day the students are given a syllabus, given a grading policy, told of the expectation of student responsibility, directed to the web-site, etc. Definitions are presented and they are sent home with some basic drill exercises. The second meeting day student presentations begin. Rather than calling on students like the Moore method [20, 22, 32, 42, 65], volunteers are requested like Cohen's modified Moore method [3].

A student receives credit for presenting regardless of the outcome of the presentation; if he is correct he must explain his work and answer questions from the class over it; if he is incorrect he is quizzed by the instructor and students as to the error or errors and is asked to amend it or consider the comments and attempt the problem again at another class meeting. There are times that another student volunteers to do the same problem after another student erred but presented and such is occassionally allowed. This is how it proceeds unless multiple people present the same problem and another had it correct; in that case the two or more are compared and discussed. Indeed, there has been at least one instance where four students presented their work (in Calculus I) such that all were correct and all solved the problem in a different manner![19]

Throughout the first few weeks the class proceeds in this fashion with short talks about new definitions, methods to prove or disprove claims, solve applied problems, and introduction to new terminology, notation, etc. By the end of the first third of the semestre, the amount of time the instructor talks decreases from perhaps half of the class period (at the end of the class session) to perhaps a fourth or not at all. In this manner, the students are encouraged to take more responsibility for their education and regard the instructor less as a teacher and more as a conductor. Nonetheless, it must be noted that some days there are no student presentations; so, the instructor must be prepared to lead a class in a discussion over some aspects of the material or be prepared to ask a series of questions that motivates the students to conjecture, hypothesise, and outline arguments that can later be rendered rigorous.

For each topic there may be some lectures but by-and-large most of the time is spent in a didactic interchange between students and the instructor, between students, or between students with some instructor guidance. When questions, hypotheses, ideas, etc. are posited by the students the instructor writes the propositions on the board (if it is not already placed there by the presenter). All are labelled as claims (until solved). Of times the claims are solved within a week or so; but, sometimes they are not. Most solutions are put forward in class, with many solved on the first attempt in lower level classes or early in a class. If a proposed solution to a claim is presented and it is in error, then the instructor may guide the students in a discussion as to correcting the error (often when the error is glaringly obvious). If the error is more severe, then the instructor may suggest

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[18]The author typically teaches Calculus I, II, or III, (of the Calculus I – IV sequence at KUP) and other courses the 100-level through 300-level (300-level and below at KUP is defined as undergraduate, 400-level mixed graduate and undergraduate, and 500-level and above graduate level).

[19]It should be noted that this was a rarity. Usually it is two or three versions correct on the same day; but it bears noting because of how enthralling the dialectic was that day.

students return to the question or may offer hints in the form of lemmas or hints toward constructing examples.

Likewise, the same is true of office interaction between the author and his students. When a student enters into the office, the interaction is oft in the form of a didactic. Whether a student presents questions, solutions, even doodles, the author attempts to pose to the student a series of questions rather than an answer. The author tries to get the student to talk about the concepts, the problems, etc. rather than the author engaging in a recitation. Whilst the student presents the author might interrupt and ask, "why," "are you sure," "is that *really* the case," etc. and the class has licence to interrupt the presenter (as well as instructor) at any juncture when a member of the class does not understand a concept, opines there is an error, etc.[20]

It is opined that these questions form a foundation for further work in mathematics and create an authentic inquiry-based learning environment. In any given class, questions are posed that are either beyond the scope of the course because of time, material, or a combination of the two. It is proposed that by leaving questions unanswered but teeming in the minds of the students, the author lays the groundwork for the student to return to the question at a later date much as we academics allow questions to fester in the recesses of our conscious or subconscious mind. In this manner and through this form of dialectic the author submits that authentic learning is engaged, true understanding might be achieved by the student, and a beginning of ac taste of what constitutes an undergraduate research experience commences.

The author's class notes for Calculus courses (indeed any course) are usually a simple set of items for discussion for the day (in some semblance of an order) that may or may not be completed on any given day since it is dependant on the students, the questions, and the didactic that is created.[21] The author's class notes simply state ideas that are to be discussed or which he anticipates will be discussed. There are occasions when students ask the instructor to go through a particular problem in depth that came from a book, the web-notes, or some other such common reference (off hand approximately once or twice a week this occurs). The author attempts the solution off the top of his head (unless the question was posed in a previous semestre; whereas the instructor lets the students know that he has solved the problem previously). Sometimes the solution is forthcoming, other times not so. In this regard the author opines that the students are allowed to see the fallibility of the instructor, see what processes he follows to solve problems, and are (hopefully) encouraged to see if the author can figure it out then it must not be too difficult, encouraged to make mistakes and to revise solutions, and encouraged not to think that everything must be solvable in a matter of seconds. This exercise of the instructor as guide is the closest thing to any semblance of group work in the author's classes and when this is undertaken it is done in such a manner that as many students as the author can involve are questioned, asked for comment, etc.

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[20]There is a sub-population of students who *hates* this. Oft such said students drop the course (and studiously avoid the author for all courses the student takes).

[21]The notes are somewhat more structured for a 300- or 400-level class because class size is often much smaller so there is less opportunity in many instances to expect most of the class time to be dominated by class discussion. However, this depends on the mathematical maturity of the class, the subject, etc. So, the discussion at the 400-level is sometimes more directed by the author and other times more directed by the students.

throughout the exercise.

Throughout the semestre, if the subject discussed is new or not in compleat continuous flow from previous material, material is oft presented after encouraging the students to read through the text first (which by-and-large they do *not* do). This encourages the students to attempt to read mathematics on their own, then the 'lecture' covers several matters regarding the material, and they are requested to re-read (which by-and-large they *do* not do since they didn't read anything in the first place) the material and attempt problems from the text, from worksheets from the course web-site, etc. For each subject, definitions, terminology, and notation are established and a series of facile claims are presented for the students to prove or disprove. It seems the case that reading is not part of the 21st century students' favourite pastimes and reading mathematics is even less popular than in the 1980s when the author was an undergraduate.

In most of the Calculus courses, the objective is to introduce the student to the subject being discussed and to allow them to work on elementary claims in the areas of math under discussion at the time and then use those elementary claims to develop more refined ideas, concepts, etc. As each course progresses, naturally, the material discussed becomes more complex, as is the case from 100 to 200 and on course levels. Expectation rises as courses are more subsequent rather than antecedent in the programme. Usually open questions posed in the Calculus courses can become the fodder for Senior Seminar projects, Directed Reading, papers for presentation at undergraduate mathematics conferences, etc. but that is not always the case. A focal point of IBL using the MMM is the instructor insists that his students (and he himself) justify every claim, every step of an argument, or every step of a solution to a problem. The word, "why," is uttered by the instructor ceaselessly throughout a class or in the office with a student or group of students.

Though this is a tangent it bears mentioning: if one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? In classical philosophy, epistemologically in order for person A to know X: (a) X must exist; (b) A must believe X; and, (c) A must justify why X is. Under our MMM we allow for (a), do not request the students adopt (b), but must *insist* on (c). This is because there are enough examples of truths in mathematical systems such that (a) and (c) are the case but (b) certainly is not for the majority. One can over time come to accept (b) because of the irrefutability of the argument that establishes the certainty of the claim.

The author is fond of quoting his late mother to the students, "mean what you say and say what you mean." The object of any lesson in the classroom is to encourage thought, to encourage deliberation, to encourage contemplation, and to encourage a healthy dose of scepticism so that one does not wander too far into a position of subservience, 'give-me-the-answer'-ism, or a position of arrogance, 'know-it-all'-ism. True IBL requires the instructor adopt an approach such that inquiry is ongoing. A demand for understanding what is and why it is, what is not known and an understanding of why it is not known, the difference between the two, and a confidence that if enough effort is exerted, then a solution can be reasoned. In this way, the MMM method is perhaps most similar to the Moore method. Consider:

Suppose someone were in a forest and he noticed some interesting things in that

forest. In looking around, he sees some animals over here, some birds over there, and so forth. Suppose someone takes his hand and says, 'Let me show you the way,' and leads him through the forest. Don't you think he has the feeling that someone took his hand and led him through there? I would rather take my time and find my own way.[22]

However, the confidence must be tempered with humility and realism. Not everything can be known. Hence, one must be selective. The instructor and students must realise that they are not the most intelligent creatures in the universe. Hence, one must accept his limitations.

The nature of conjectures arising from the students is in keeping with the classic Moore method. However, as with the traditional method, an instructor employing the MMM in an IBL Calculus class is free (and indeed should) pose pertinent questions to students which might not germinate from the students. Oft this *is* dependent on the nature of the composition of the student body taking the class in a particular semestre. Thus, again it should be noted that the MMM method requires as much flexibility as possible on the part of the instructor to gauge the mathematical maturity of the class members and adjust (not dumb-down or treat content differently dependent on the audience) accordingly. Each semester brings with it new students and so new challenges. It is incumbent upon the instructor to keep vigil and assess the progress of the students.

The author found that in his previous situation at Morehouse College (MC) which followed a traditional Calculus I – II – III three four-credit semester courses it was much easier to incorporate authentic IBL questions in the classes for the schedule placed the course at an hour over five days and the author used all five days (so he had a 'bonus' day per week which allowed for much more discussion) and because there was, generally, a higher accord for mathematics and the Mathematics Department at Morehouse that at his present situation. It has been very difficult, to say the least, at Kutztown University of Pennsylvania (KUP) where there exists Calculus I – II – III - IV of 3 semester credits each.

It is very difficult to incorporate authentic IBL questions at KUP because about half the Calculus classes are taught in a two day a week $1\frac{1}{2}$ hour block schedule and it seems that a superfluous understanding of a topic is all that can be achieved in many cases. The author may be wrong but such limited interaction between students and instructors does not seem ideal nor educationally worthy. There is little or no extra time as there was at MC to delve into a tangential topic or even a main topic with any gusto. The scheduling of classes at KUP is very much structured counter to an IBL environment and more the structure is more attuned to a traditional recitation German-seminar method or calculator-based applied constructivist method. The difficulty in incorporating IBL questions in the Calculus is further compounded by, it seems, the nature of the university, college, department, and students. The students and faculty expect a class to begin and end precisely on time - - no deviation whatsoever. Therefore any discussion is oft interrupted by class ending abruptly (which is to say students packing up and leaving or preparing to leave five minutes before the end of class). Furthermore, there is a distinct and unmistakable 'flavour' of teacher-education in the programme at KUP which seems

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[22]R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966), videocassette

to centre on material should be understood or discussed only if it is that which a pre-service teacher will teach when that student is a faculty member at some school. Also, there is a dearth of science, technology, engineering, or mathematics (STEM) majors at KUP versus MC and many of the STEM programmes are astonishing lacking in mathematics requirements for the major and, perhaps, standards.[23] The department is not situated in its own building or floor so there might be a History class immediately before the Calculus class and an English class after the Calculus class. Thus, any semblance of an *esprit d' corps* on the part of the faculty or the students is lacking.

To say that incorporating inquiry-based learning methods under a prescript of the modified Moore method is a challenge is an understatement for a faculty member at Kutztown University of Pennsylvania. Nonetheless, there is support from the author's colleagues for its use in Calculus and in other areas of mathematics. That such support exists is not testament to the author but to the success that the Moore method and its derivatives seem to produce. So, there is both the potential for success or failure (as with any endeavour) when one seeks to incorporate inquiry-based learning methods under a prescript of the modified Moore method in Calculus.

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[23]Two examples illustrates the point. A major in Computer Science at Morehouse requires Calculus I, II, II, Linear Algebra, Discrete Mathematics, and Probability & Statistics. One track for a major in Computer Science at KUP requires Discrete Mathematics and 'Calculus for Business or Information Sciences' only. A major in Biology at Morehouse requires Calculus I and Calculus II. A major in Biology at KUP requires Trigonometry and Applied Statistics (non-calculus based introductory).

IV. Topics or Materials where Opportunity Exists for Authentic
Inquiry-Based Leaning to Flourish in the Calculus Courses.

So, a course in the Calculus sequence is arguably the most difficult course
in which inquiry-based learning (IBL) can be achieved with any degree of success
within the curriculum especially because of the 'coverage' consideration since if a
critical concept or area was not 'covered' in Calculus I or II it might do serious
harm to the student in Calculus II, III, or beyond where the understanding the
topic may depend significantly on said material which was not 'covered.' We shall
use as our reference the Calculus I – IV model employed at KUP and restrict the
discussion to Calculus I – III since the author has not had the opportunity to teach
Calculus IV at KUP as of the writing of this paper.

The author uses a system of highlighting important principles of a topic in Cal-
culus, or any class, by including materials for the students on his web-site. The
web-site is an ever changing and evolving collection of author created handouts,
worksheets, and reaction papers to student work (usually on tests or quizzes) and
includes some student presentations (scanned then uploaded as *.pdf files) that are
of interest.

Each class Calculus I, Calculus II, and Calculus III includes links of import to
handouts or materials from previous courses or a set of reviews for material pre-
requisite to Calculus I. Some of the topics where an opportunity exists for authentic
inquiry-based learning to flourish are not of Calculus, per se, but of pre-requisite
material or tangential material. Much of it delves into the Topology of $\mathbb{R}$ or other
ideas from Real Analysis. It must be restated that many claims *come from the
students* and as such differ from semester to semester. Also it is very important
to note that not all questions posed by instructor or students are answered in a
course.[24]

In Calculus I the 'coverage' is:

0. Preliminaries
A brief discussion of pre-calculus topics of import.

1. Limits and Continuity
Intuitive concept of limit; definition of limit; theorems on limits, calculation of
limits, limits at infinity or infinite limits; definition of continuity; theorems on con-
tinuity; and, the squeeze theorem.

2. Differentiation
The definition of the derivative; geometric interpretation of derivatives; the power
functions and their derivatives; theorems on differentiation; the chain rule and its

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[24]If the question is of import such that leaving it 'hanging' would do harm to the students,
then it is not left alone but is answered in a timely manner (if the students cannot come to a
solution the author gives a quiz usually which creates the solution (breaks the question into small
parts with many hints to allow students to deduce a solution)). If no student gets the solution,
then and only then will the author supply a solution. If the question can be left, then it is for the
good of the students so they might also understand that not everything is solved and not every
mathematics problem is done in five minutes or less (one would be surprised at how prevalent
such an idea is amongst the student body).

applications; instantaneous velocity, speed, acceleration; implicit differentiation; the differential and approximation; differential notation; and, related rates.

3. Applications of the Derivative
Extrema of functions; the mean value theorem; first derivative test; concavity and second derivative test; optimization problems; rectilinear motion and other applications; and, Newton's method.

If time permits any discussion of anti-derivatives, indefinite integration, or definite integration is welcomed but not expected.

The author has found that some very nice problems, claims, and discussions arise in each part of the course. For preliminaries, we usually discuss basic algebra, analytic geometry, and functions in the plane and rigorously remind (or introduce) students of the domain, codomain, and range of a function; invertibility of some functions; and claims about functions. For example:

1. Claim: A function in the plane has a vertical asymptote at x = a if and only the function does not exist at x = a.

2. Claim: Let x ∈ ℝ,

$$\frac{x^3 - 1}{x - 1} = x^2$$

3. Claim: Let x ∈ ℝ,

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

4. Claim: Let x ∈ ℝ,

$$\frac{x^{-2}y^{-2}}{x^{-2} - y^{-2}} = \frac{x^2 - y^2}{x^2 y^2}$$

5. Claim: A function in the plane is invertible if and only if it is injective (in the text used at KUP this is stated as true!).

6. Claim: Let x ∈ ℝ, $\sin^2(x) + \cos^2(x) = 1 \quad \Rightarrow \quad \sin(x) + \cos(x) = 1$.

7. Claim: $0.\overline{9} < 1$.

8. Claim: $0 < 1$.

9. Claim: $x \cdot 0 = 0$.

10. Claim: $x \div 0 = 0$

By and large rather facile claims but ones which focus the student's attention on points of import for later in the course.

For the discussion of limits we discuss limits at a point in terms of the intuitive left or right limits (computational) and present the formal definition but do not use it. For the discussion of continuity we discuss continuity at a point and build from the limits at a point in terms of the intuitive left or right limits (computational) along with the function value at the point. We present the formal definition but do not use it. For the theorems on limits, calculation of limits, and theorems on continuity we work on said but we do not accentuate any of the topics usually.

However, the topics of continuity at a point, limits at infinity or infinite limits and the squeeze theorem oft supply many interesting questions for discussion and inquiry-based exercises.

Some typical questions for discussion and claims from this material usually include:

1. Question: Let f be a well defined real-valued function[25] in the plane with domain D. Let x = a be in D. What does it mean for the limit at $x = a$ to not exist?

2. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is continuous everywhere in D?

3. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is continuous nowhere in D?

4. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is continuous at exactly one value in D?

5. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is continuous at infinitely many values in D and is not continuous at infinitely many values in D?

6. Question: Let f be a well defined real-valued function in the plane with domain D. Let x = a be in D. Can there exist a function in the plane where the limit as x approaches x =a is ∞?

7. Question: Let f be a well defined real-valued function in the plane with domain D where there exists a $k \in \mathbb{R}$ such that $(k, \infty) \subseteq D$. What does $\lim_{x \to \infty} f(x)$ mean?

8. Question: Let f be a well defined real-valued function in the plane with domain D where there exists a $k \in \mathbb{R}$ such that $(k, \infty) \subseteq D$. Suppose $\lim_{x \to \infty} f(x)$ exists, does that imply $\lim_{x \to -\infty} f(x)$ exists?

9. What is a vertical asymptote; how does said relate to limits?

10. What is a horizontal asymptote; how does said relate to limits?

11. What is an oblique asymptote; how does said relate to limits?

12. Question: Let f , g, and h be a well defined real-valued functions in the plane with domain D and both $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ exist. What does that imply about h and how?

When we move to the topic of differentiation that which receives special attention includes the definition of the derivative, implicit differentiation, and the differential. The three contrast nicely to help student understand the concepts of a variable, an independent variable, and a dependent variable.

Some typical questions or claims for discussion from this material usually include:

1. A plethora of questions requesting student find $f'(x)$ using the $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ definition when such exists.

2. Question: Let f be a well defined real-valued function in the plane with domain D.
Let $(a, b) \subseteq D$ and $c \in (a, b)$. What does $f'(c)$ mean? How is it found considering $f'(x)$ using the $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ definition and evaluating at $x = c$, considering $f'(c)$ using the $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ definition, versus considering $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ where such (if any) exist?

3. Question: Let f be a well defined real-valued function in the plane with domain D. Let $(a, b) \subseteq D$ and $c \in (a, b)$. Does existence imply continuity? Does existence imply differentiability? Does continuity imply existence? Does differentiability imply existence? Does differentiability imply continuity? Does continuity

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[25]Obviously herein we mean that $f : D \longrightarrow \mathbb{R}$ such that $D \subseteq \mathbb{R}$ is a well defined function.

imply differentiability? Does non-differentiability imply non-continuity? Does non-continuity imply non-differentiability?[26]

4. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is continuous everywhere in D but not differentiable everywhere in D?

5. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is continuous at a point in D but not differentiable at that point in D?

6. Question: Let f be a well defined real-valued function in the plane with domain D. Does there exist a function in the plane that is differentiable at a point in D but not continuous at that point in D?

7. Claim: Let f be a well defined real-valued function in the plane with domain D. Let (a, b) $\subseteq$ D and c $\in$ (a, b). Suppose $\exists$ a well defined tangent line to f at (c, f(c)), then there must exist a well defined normal line to f at (c, f(c)).

8. Question: Let E be a well defined ellipse in the plane. Does there exist a function in the plane that is E? What is the implicit derivative of E with respect to x, implicit derivative of E with respect to y, implicit derivative of E with respect to t where t is time? Is there a group of functions in the plane whose union is E? What are the derivatives of those functions and how do they relate to the implicit derivative of E?

One exchange from a class in Spring of 2007 is illustrative of the fun that occurs in this part of the course. The students are asked to find $f'(x)$ using the

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

definition where $f : \mathbb{R} \to \mathbb{R} \quad \ni f(x) = e^x$.

The author asks for a volunteer to do the problem at the board. There was silence in the class and finally one particularly sharp student said she would volunteer. After she went to the board and reached an impasse, the author asked, "what if I were to give you the answer?" The student replied, "Then we would be happy." The class erupted in laughter as did the author. The author replied, "no." He asked about some pre-calculus concepts, namely the distributive property of multiplication over addition, the law of exponents, and the graphs of certain functions (graphs without calculators). He guided her to $a, b, c \in \mathbb{R}$

$$a \cdot b + a \cdot c = a(b+c), \quad a^{b \cdot} \cdot a^c = a^{b+c}, \quad \wedge \quad a^{b \cdot c} = \left(a^b\right)^c.$$

He further asked them to consider

$$f_1 : \mathbb{R} \to \mathbb{R} \quad \ni f_1(x) = e^x,$$
$$f_2 : \mathbb{R} \to \mathbb{R} \quad \ni f_2(x) = e^x - 1,$$
$$f_3 : \mathbb{R} \to \mathbb{R} \quad \ni f_3(x) = x,$$
$$f_4 : ((-\infty) \cup (0, \infty)) \to \mathbb{R} \quad \ni f_4(x) = \frac{e^x - 1}{x}.$$

especially the graphs of said and how each might relate to the question left 'hanging.' The class ended leaving the volunteer and others to ponder the hint and problem. When they returned, as I recall, she explained why $f(x) = e^x \Rightarrow \quad f'(x) = e^x$. It

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[26]These questions are fun for they allow the student to conceptualise statements of theorems and the contrapositive, converse, and inverse of a theorem.

gave rise to the definition of functions whose derivatives are cyclic and:
Question: Let f be a well defined function in the plane with domain D. Let (a, b) $\subseteq$ D. Does there exist a function in the plane that where $f'(x) \neq f(x)$ but $f''(x) = f(x)$? Does there exist a function in the plane that where $f'(x) \neq f(x)$, $f''(x) \neq f(x)$, but $f'''(x) = f(x)$ (and inductively for higher order derivatives) ?

When Calculus I continues to applications of the derivative there is not as much authentic inquiry-based[27] questions posed, inquiries about, or discussions arising. It may be due to the nature of the section of the course; after all it is concerned with applications such as extrema of functions; concavity; graphing; optimization problems; rectilinear motion and other applications; and, Newton's method. The author spends little to no time on Newton's method, enjoys contrasting applied extrema problems to related rate problems, and accentuates graphing using positive-negative analysis of the first and second derivatives to allow students to review several pre-calculus topics including the Rational Root Theorem, Descartes Law of Sign Changes, the Fundamental Theorem of Algebra, synthetic division and long division of polynomials, etc.

However, there are some opportunities for authentic IBL exercises to be considered in this section of Calculus I, for example, with the mean value theorem the author does request student to ponder how such relates to Rolle's Theorem. Also, L'Hôpital's Rule the students are constantly questioned (some would say berated) about the hypotheses of L'Hôpital's Rule and forms of limits to which L'Hôpital's Rule may be applied. We generalise the discussion to include questions on the meaning or significance of "∞" + "∞," "∞" − "∞," "∞" ÷ "∞," $0 \div 0$, $0 \cdot$ "∞," "0^{∞}," "1^{∞}," "∞^{0}," etc. If time permits any discussion of anti-derivatives, indefinite integration, or definite integration the author approaches said completely as an IBL opportunity for it overlaps Calculus II and is a time permits optional section of the course.

In Calculus II the 'coverage' is:

0. Preliminaries
A brief discussion of pre-Calculus II topics of import.

1. The Integral
Anti-derivatives and indefinite integration; change of variables; summation notation and area; the definite integral; properties of the definite integral; the Fundamental Theorem of Calculus (parts I and II); integration by substitution; numerical integration; and the logarithm defined as an integral.

2. Applications of the Definite Integral
Area of a region in a plane; volume of a solid of revolution (shell, disc, geometric methods); surface area of a solid of revolution; average value of a function; work; fluid pressure; arc length; and, other applications

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[27]We mean inquiry-based learning (IBL) in the sense of opining, conjecturing, showing claims true or false, etc. about theoretical or 'pure' mathematics not in the way others might define it so as to include applied projects, use of Maple, Mathematica, etc. Those activities are a part of the author's Calculus classes but are not considered authentic IBL activities by the author because there is too much witnessing of graphs, ease of regurgitation of a syntax to create a result, etc. .

3. Principles of Integration

Integration by parts; trigonometric integrals; integration by trigonometric substitution; integration with partial fraction decomposition; integration by other forms of substitution; Trapezoid and Simpson's rules approximations; integration with Maple and approximations; and improper integrals.

If time permits a discussion of Differential Equations (1st order DEs, Euler's Method, 1st order DE applications and models; or, 2nd order linear homogeneous DEs) is suggested.

The author has found that some very nice problems, claims, and discussions arise in each part of this course. For preliminaries, we usually discuss basic algebra, analytic geometry, functions in the plane, limits and continuity; differentiation (especially the definition of the derivative and the different methods of differentiation); and a few applications of the derivative. It is typically the case that the claims (which were a part of Calculus I) of $0 < 1$, $x \cdot 0 = 0$, and $0.\overline{9} = 1^{28}$ are the first discussed.

For the first part of actual Calculus II content that is stressed includes antiderivatives and indefinite integration; summation notation and the Riemann integral; and the Fundamental Theorem of Calculus (parts I and II). Content that is not accentuated as heavily includes integration by substitution and change of variables; numerical integration; the logarithm defined as an integral; and, properties of the definite integral.

Of the material stressed we approach the definition of antiderivatives from the standpoint of 'opposite' of derivative and from the perspective of differential equations. The author asks students to opine given $D \subseteq \mathbb{R}$ and $f : D \longrightarrow \mathbb{R}$ is a well defined real-valued function whilst $g : D \longrightarrow \mathbb{R}$ is also a well defined real-valued function and furthermore $f'(\text{x})$ and $g'(\text{x})$ exists for all $x \in (a,b) \ni (a,b) \subseteq D$ what are anti-derivatives for functions such as $h(x) = f(x) + g(x)$, $h(x) = f(x) \cdot g(x)$, $h(x) = f(x) \div g(x)$, where $g(\text{x}) \neq 0$, or $h(x) = (f \circ g)(x)$.

Inquiry-based learning begins at this point if not before for Calculus II. For Riemann sums the author insists that students compleat arguments as to the area of a region R bounded by $x = a, x = b \ni a < b, y = 0 \wedge y = f(x) \ni f(x)$ is a polynomial function of degree 3 or less. That predicates a discussion of mathematical induction and arguments about summations precedes the Riemann sums. Also, we stress the Fundamental Theorem of Calculus (parts I and II) over and over once the theorem is introduced and attempt to link both parts to that which was learnt in at the point of the discussion back to birth.

Typical questions for discussion and claims from this material usually include:

1. Question: Let f be a well defined real-valued function in the plane with domain D and let $(a,b) \subseteq D$. Must $\int f(x)\,dx$ exist? Must $\int_{a}^{b} f(x)\,dx$ exist?

2. Question: Let f be a well defined real-valued function in the plane with domain D and $f'(x)$ exist $\forall x \in (a,b) \ni (a,b) \subseteq D$. Must $\int f'(x)\,dx$ exist? Must $\int_{a}^{b} f'(x)\,dx$ exist?

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[28]Note the claim by Calculus II is equality rather than appealing to what the students want but what is not true.

3. Question: Let f be a well defined real-valued function in the plane with domain D and g be a well defined real-valued function in the plane with domain D and $(a, b) \subseteq D$. What conditions might be necessary or sufficient for $\int (f(x) + g(x))\, dx$ to exist?

4. Question: Let f be a well defined real-valued function in the plane with domain D and g be a well defined real-valued function in the plane with domain D and $(a, b) \subseteq D$. What conditions might be necessary or sufficient for $\int (f(x) \cdot g(x))\, dx$ to exist?

5. Question: Let f be a well defined real-valued function in the plane with domain D and g be a well defined real-valued function in the plane with domain D and $(a, b) \subseteq D$. What conditions might be necessary or sufficient for $\int (f(x) \div g(x))\, dx \ni g(x) \neq 0$ to exist?

6. After realising the product rule or quotient rule do not have 'clean' analogues in Calculus II, the question is posed, do there exist functions $f \wedge g$ (well defined real-valued functions in the plane) with domain $D \wedge (a, b) \subseteq D \ni$

$$\int_a^b (f(x) \cdot g(x))\, dx = \left(\int_a^b (f(x))\, dx \cdot \int_a^b (g(x))\, dx \right)$$

and can they construct said so that the claim is non-trivial?[29]

For the section of the course on applications of the definite integral emphasis is placed on area of regions in a plane; volume of a solid of revolution; surface area of a solid of revolution; average value of a function; and, arc length. Very little time is devoted to the other topics in this section. Moreover, much emphasis is placed on attempting to assist the student in linking the concept of the Riemann sum with the definite integral.[30] The type of IBL in this section does differ from other sections in the Calculus sequence insofar as the accent is on problem-solving and getting the students to **do** many different types of area problems; volume of solids of revolution problems; surface area of a solid of revolution problems; and, arc length problems. The accent is on doing the problems without machines and to allow the students to conjecture and opine. Out objective is to get the student thinking inductively so that they may conjecture and hypothesise whilst doing what is considered drill exercises by some.

It is worth noting at this point that drill exercises, memorisation, and traditional paper-and-pencil problems are included in the author's courses. The author opines that such is not contrary to authentic inquiry-based learning; in fact, we opine it is more 'realistic' than many of the reform exercises where every problem requires a computer to 'solve' the problem (solve or produce an approximation) for the author has found many students lack the ability to differentiate between solutions or approximations. Furthermore, the CUPM (2004) guidelines warn of technology being overused and becoming a 'crutch'[31] and the author has found many students are not challenged to comprehend that which is needed to be memorised versus memorisation without justification.[32]

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[29]This is fun for it assists the students in thinking about the difference between and betwixt universals and existentials for their later work in mathematics.

[30]Rather than just handling as something disjoint or skipping Riemann sums as some instructors do.

[31]See page 24, CUPM Curriculum Guide 2004, Washington, DC, MAA, 2004.

[32]We opine that the use of technology is fine *after* a person learns the 'how' and 'why' of a mathematical principle but there is too much of it being used today to 'skip over' the 'hard' part of

By the latter part of Calculus II, content that is accentuated in the part of the course includes integration by parts (IBP); trigonometric integrals; trigonometric substitution; partial fraction decomposition; and, improper integrals. Material not skipped but not stressed as highly includes integration by other forms of substitution; Trapezoid and Simpson's rules; integration with Maple; and, approximations.

The author enjoys introducing the Gamma function to students in the course during the discussion of IBP and asks them to dwell on questions such as finding $\Gamma(k) \ni k \in \mathbb{N}^*$,[33] presenting then with the claim that $\Gamma(k) = (k-1) \cdot \Gamma(k-1)$ when $k \in \mathbb{N}$, etc. which oft produces some interesting claims on the part of the students and lead at least one student at KUP and two students at MC to do a Senior Seminar thesis on the Gamma function. Also, for integration by parts the connection to the chain rule for derivative is stressed and the author again asks the students for some sort of opining as to which rules of derivatives 'reverse' well (we ask for creation of some anti-derivative rules again). The topic of partial fraction decomposition allows for much discussion of factorisation, the Fundamental Theorem of Arithmetic, the Fundamental Theorem of Algebra, solutions to systems of linear equations, etc. The topics of trigonometric integrals and trigonometric substitution allows for much discussion of basic unit circle or functional trigonometry and the use or radian measure; gets the students to hark back to some fundamental trigonometric identities, etc. Also, with the Trapezoid or Simpson's rules approximations the author uses said to introduce the notion of non-Riemann integrability.

For improper integrals the author forces a rigorous discussion of the Fundamental Theorem of Calculus (part I) and stresses why limits must be employed in order to solve improper integrals. In this way it allows for the students to concentrate of the nuances of the notation used in Calculus as well as the stunning ideas of some consequences of improper integrals. Inevitably, a student or a few students reject the idea and demonstrate disgust with Gabriel's horn[34] having volume but the graph it is created from having no area. Recall that the area of the region, R, bounded by $y = 0, x = 1, y = \frac{1}{x}$, to the right of $x = 1$ does not exist since

$$\int_1^\infty \frac{1}{x} dx$$

does not exist. Yet, the volume of the resulting object, H (Gabriel's Horn), obtained by rotating R about the x-axis does exist since

$$\int_1^\infty \frac{\pi}{x^2} dx$$

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some problems and that instructors at the pre-college level are either forced or strongly encouraged to use calculators at every turn either because of National Council of Teachers of Mathematics (NCTM) Standards (better know as the document endorsing the dumbing down of pre-college mathematics) or because of the pressure to pass students, 'skip over' the 'hard material,' or perhaps because if an instructor has been teaching the same material for over a quarter-century that instructor might just be getting a tad bored and the calculators or computers relieve the drudgery of the job (i.e.: it is something *new* and interesting).

[33]We use $\mathbb{N}^*$ to mean the cardinal naturals and $\mathbb{N}$ to mean the ordinal naturals in the author's classes.

[34]The region, R, bounded by $y = 0, y = \frac{1}{x}, x = 1$, to the right of $x = 1$ then spun about the x-axis is, I believe, standardly called Gabriel's horn.

exists. Hence the region R has no area (i.e.: $\nexists a \in \mathbb{R} \ni R$ has a units^2 area) but the region H (created from R) has volume (π units^3)!

Gabriel's horn provides so much rich material for discussion and much of it for students arises from scepticism: not accepting that which they see is. Gabriel's horn conflicts with their intuition and its existence usually sparks 'an open revolt' in the class. Unfortunately, some students retreat to the book and accept it because it is so written, but the author tries to discourage this behaviour. The author tries to encourage the students to open the mind and leave the desire to make math work as one wants rather than how it is and see the joy of the non-intuitive. The author is not always successful, but he tries (and as of this writing has not given up).

So, questions include:

1. Question: Let R be a well defined region in the plane. Does existence of an area imply existence of a volume of solid of revolution about the x-axis or y-axis?[35]

2. Question: Let R be a well defined region in the plane. Does existence of a surface area imply existence of a volume of solid of revolution about the x-axis or y-axis?

3. Question: Let R be a well defined region in the plane. Does existence of a volume of solid of revolution about the x-axis or y-axis imply existence of an area?

4. Question: Let R be a well defined region in the plane. Does existence of an arc length imply existence of an area, volume of solid of revolution about the x-axis or y-axis, or surface area of a solid (other variations of the permutations of the concepts are also posed) ?

5. Question: Let f be a well defined real-valued function in the plane with domain D and let $(a,b) \subseteq D, c \in (a,b)$. Is $\int_{c}^{x} f(t)\,dt$ always well-defined? Must $\int_{a}^{x} f(t)\,dt$ exist?

As stated earlier, If time permits a discussion of Differential Equations is suggested; but, the author has only reached the point of 1st order DEs so little is accomplished in an IBL or regular sense.

In Calculus III the 'coverage' is:

0. Preliminaries
A brief discussion of pre-Calculus III topics of import.

1. Indeterminate forms for limits and improper integrals.
The special types of limits and integrals constitute the first section of the course (which overlaps Calculus I and Calculus II). The discussion includes basic indeterminate forms and other indeterminate forms of limits and then considers improper integrals with infinite limits of integration and improper integrals with discontinuous integrands.

2. Infinite Sequences and Series.
The class focus turns to infinite sequences; convergent or divergent series; infinite

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[35]Existence implies that the length, area, or volume is $b \ni b \in \mathbb{R}$ not the nonsensical length, area, or volume is ∞.

series of positive terms; the integral test, the ratio test, and root test; alternating series and absolute convergence; power series and function representation; differentiation and integration of power series; Taylor and Maclaurin series; applications of Taylor polynomials; and, the binomial series.

3. Conic Sections
The sections of this part of the course include standard conic sections: parabolas, ellipses; hyperbolas; and, rotation of axes is optional.

4. Plane Curves and Polar Coordinates
The class focus turns to plane curves; tangent lines and arc length; polar coordinates; derivatives and integrals in polar coordinates.

5. Vectors and Vector-valued Functions
The last topic section overlaps material in Calculus IV and is optional. The topics are vectors in two and three dimensions; dot product; vector product; lines and planes; vector valued functions and space curves, and limits, derivatives, and integrals.

Calculus III at Kutztown University of Pennsylvania (KUP) essentially wraps up material that is in Calculus I – II at most universities that employ a traditional Calculus I – II – III three four-credit semester courses (which is not surprising since in that system there are 8 semester hours and in the KUP system there are nine semester hours; but, it should be noted that implies that multivariate Calculus is treated at a traditional institution with four semester hours whereas at KUP it is but three semester hours).

The Calculus III seems a 'chopped up' course with many topics but it's focus is truly on infinite sequences and series and polar coordinates. The special types of limits and integrals constitute the first section of the course the author treats such as review but it is a bit longer review than a normal review— it lasts approximately a week and some students have great difficulty with the material if they had Calculus I and II where all problems were exercises and were of a form such as, "find $\frac{dy}{dx}$ where $y = e^x \cos(x)$ using the product rule,"[36] "find $\int_{3}^{7} \frac{5}{1+x^2} dx$ using the substitution x = $\tan(\theta)$," "find $\frac{dy}{dx}$ when $x = 0$ and $y = 8$ for $x^2 + x - e^x \cdot \cos(1) + y = 8$ using Maple,"[37] or " A box's sides are growing in such a manner that the length is shrinking at a rate of 2 inches per minute, the width is growing at a rate of 3 inches per minute, and the height is increasing at 4 in/min. Find how fast the volume of the box is changing when it is a cube of volume 1728 in^3 using your TI-83 and change the values of x and y to explore this idea on your TI-83. Graph it using your TI-83."[38] Such students, in the three years the author has been at KUP, have dropped the author's course section.

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[36]And the product rule is on an instructor approved 'crypt sheet!'

[37]For some classes the Maple code is provided.

[38]For at least one class the instructor writes the TI-83 programmes and has the students come to the instructor's office to download the programme onto the student's calculator. If the instructor opines that the programming is not mathematics, then I am left confused and wonder exactly where is the mathematics in this exercise?

After the preliminaries and the review of the special types of limits and integrals focus shifts to infinite sequences and series. The author weaves a discussion of sequences throughout the discussion of series because of sequences of partial sums, the theorem which nicely gives the result that $\sum_{i=1}^{\infty} a_i$ does not converge if the sequence of the terms does not converge to zero, and because he has found that at MC and KUP students have a very hard time distinguishing between and betwixt sequences and series. All of the topics are discussed in more than a superfluous way. The entire section of the course is repleat with 'meaty' ideas and questions to provoke IBL in the students.

So, questions include:

1. Question: Show it is the case or it is not that $\{g(k)\}_{k=1}^{\infty}$ is monotonic and bounded (for different expressions, $g(k)$).

2. Question: Show it is the case or it is not that if $\{g(k)\}_{k=1}^{\infty}$ is monotonic and not bounded it is convergent.

3. Question: Show it is the case or it is not that if $\{g(k)\}_{k=1}^{\infty}$ is monotonic and not bounded it is divergent.

4. Question: Show it is the case or it is not that if $\{g(k)\}_{k=1}^{\infty}$ is bounded and not monotonic it is convergent.

5. Question: Show it is the case or it is not that if $\{g(k)\}_{k=1}^{\infty}$ is bounded and not monotonic it is divergent.

6. Claim: there exists a sequence, $\{g(k)\}_{k=1}^{\infty}$, that is both convergent and divergent.

7. Question: Suppose $\sum_{k=1}^{\infty} g(k)$ is convergent and let $g(x)$ be the algebraic idempotent extension function of $\{g(k)\}_{k=1}^{\infty}$ for domain $[1, \infty)$, must it be the case that $\int_{1}^{\infty} g(x)dx$ exists?

8. Question: Suppose $g(x)$ be is a well defined real-valued function from domain $[1, \infty)$ to $\mathbb{R}$, $\int_{1}^{\infty} g(x)dx$ exists, and the restriction function of $g(x)$ restricted to $\mathbb{N}$ is $\{g(k)\}_{k=1}^{\infty}$, must it be the case that $\sum_{k=1}^{\infty} g(k)$ is convergent?

9. Question: Show it is the case or it is not that $\sum_{k=1}^{\infty} g(k)$ is convergent (for different expressions, $g(k)$) using a direct comparison test. Explain why you the chose the series you choose for a comparison and justify each step of the argument.

10. Question[39]: Show it is the case or it is not that $\sum_{k=1}^{\infty} g(k)$ is a divergent series and $\sum_{k=1}^{\infty} h(k)$ is a divergent series but then $\sum_{k=1}^{\infty} (g(k) + h(k))$ is a convergent series.

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[39]This exercise suggests partial fraction decomposition for telescoping series and, I opine, assists students in connecting the telescoping series work to the concept of "∞" + "-∞" from Calculus I and II.

11. Question[40]: Show it is the case or it is not that $\sum\limits_{k=1}^{\infty} g(k)$ is a convergent alter-

nating series, then $\sum\limits_{k=1}^{\infty} |g(k)|$ converges.

12. Question: Show it is the case or it is not that $\sum\limits_{k=1}^{\infty} |g(k)|$ converges, then $\sum\limits_{k=1}^{\infty} g(k)$ is a convergent alternating series.

13. Question: Show it is the case or it is not that $\sum\limits_{k=1}^{\infty} g(k)$ is a divergent alternating

series, then $\sum\limits_{k=1}^{\infty} |g(k)|$ diverges.

14. Question: Show it is the case or it is not that $\sum\limits_{k=1}^{\infty} |g(k)|$ diverges, then $\sum\limits_{k=1}^{\infty} g(k)$ is a divergent alternating series.

15. Question: Let f be a well defined real-valued function in the plane with domain D and let (a, b) $\subseteq D$. Must there exist a Maclaurin series for f ? Must there exist a Taylor series for f at c where $c \in (a, b)$?

16. Question: Are there any properties for a well defined real-valued function in the plane, f, with domain D and (a, b) $\subseteq$ D for a Maclaurin series for f to exist or for a Taylor series for f to exist at c where $c \in (a, b)$?

17. Question: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = e^{-x^2}$ create a Maclaurin polynomial of degree 12 for f and use it to approximate $e^{-(3)^2}$. Explain why such an approximation is not $e^{-(3)^2}$.

18. Question[41]: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = e^{-x^2}$. What is the least degree of a Maclaurin polynomial in order to approximate $e^{-(0.1)^2}$ within an error bound of 10^{-5}.

19. Question[42]: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = e^{-x^2}$ create a Maclaurin polynomial of degree 12 for f and use it to approximate $\int\limits_{-2}^{3} f(x)dx$. Explain why such an ap-

proximation exists. Find $\int\limits_{-2}^{3} f(x)dx$.

It seems that the material of this section dominates about half the semester so that some (more than trite) discussion can ensue about infinite sequences; convergent or divergent series; infinite series of positive terms; the integral test, the ratio test, and root test; alternating series and absolute convergence; power series and function representation; differentiation and integration of power series; Taylor and Maclaurin series; and, applications of Taylor polynomials. We have not even mentioned the investigation of radii of convergence and intervals of convergence for power series or the creation of power series from standard Maclaurin series (e.g.: a power series for $f(x) = e^x \sin(x)$ from the Maclaurin series for e^x and $\sin(x)$)!

Then in only one day conic sections are mentioned. This is because usually a

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[40]Questions 10 – 13 seem to assist students in better understanding absolute convergence, conditional convergence, and divergence.

[41]This question allows student to discover that if the series is alternating it is easier to use the alternating series approximation theorem rather than the Taylor Remainder Theorem.

[42]They should, of course, come in with such a definite integral cannot be found. This exercise, I have found, helps student to see that when a problem is termed find that does not mean it can be found (e.g.: not everything exists).

third or more of the class had a treatment of it in High School, it is an opportunity to leave the students to actually read some math and attempt it without direction,[43] Attention immediately diverts to plane curves and polar coordinates - - plane curves; tangent lines and arc length; polar coordinates; derivatives and integrals in polar coordinates.

The authentic IBL is a part of this section as well for *most* students had not worked with polar coordinates so it allows for some wonderful discussion about: polar versus Cartesian coordinates – which graphs are better expressed in which system; the non-uniqueness of abscissas and ordinates for points in polar; the conversion of polar to Cartesian coordinates and visa versa; finding equations of tangent lines to polar graphs (a delight!); arc length; derivatives; and, the wonderful integrals in polar coordinates. A traditional exposition is not the way the author approaches getting students to realise that area is not the same expression in polar as it is in Cartesian. Under the author's MMM, after the discussion of polar coordinates, graphs, and derivative he queries the class about area of graph in the polar system.

The class usually opines to do polar area as with the x-y plane system. So, a student volunteers to, let us say, find the area of r $= 1 + \cos(\theta)$, $\theta \in [0, \pi]$ let us say. The student claims the area is $\int_0^\pi ((1 + \cos(\theta)) - 0)d\theta$, the student compleats the problem and the students are satisfied. The author give the class a worksheet where one of the problems is of the form find the area of $f(\theta) = r = \cos(\theta)$. When the class returns the next period it is fair to say there is 'chaos' for the students are loathe to confront the reality that the graph is a circle and the geometric process of finding the area yields a different result from the $\int_0^{2\pi} (f(\theta) - 0)d\theta$. A didactic discussion allows for the class as a whole to work with the author and produce the classic $\int_{\theta_1}^{\theta_2} \frac{1}{2}(f(\theta))^2 d\theta$ for the area of polar figure bounded by r $= f(\theta)$ from r $= \theta_1$ to r $= \theta_2$ where $f(\theta)0 \quad \forall \quad \theta \in (\theta_1, \theta_2)$.

At least a third of the semester is spent on polar graphs and parametric equations; the parametric equations offer a nice reminder and refresher to implicit derivatives. Also, the author can frankly point out that he has never reached the Vectors and Vector-valued Functions section in Calculus III whilst at KUP.

The overlaps between courses at KUP should become apparent to the reader. It is not for the author to say that such overlap is pedagogically sound or not; but he does opine that there is much overlap throughout the mathematics programme. Such also existed at MC but there the programme was revised in 2002 and much overlap deleted. The author does not know if the deletion of much of the overlap at MC was successful or not. Perhaps it is too early to tell (a ten year period would suffice one assumes). It is of interest, though, that the KUP mathematics programme is much more redundant than that at MC.

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[43]The students typically do not do the assignment once they discover that they are not tested over the material. It is understandable, but nonetheless sad.

V. Successes and Lack thereof.

It probably comes as no surprise that there have been some set-backs along the way; but, there were also successes.

It seems that through the years there are more successes at the Calculus I –level or Calculus II –level than Calculus III –level for a standard I – III sequence. For the I – IV sequence we cannot say with any credulity but the seems anecdotal evidence that the earlier, the better (not surprising). The earlier a student enrols in the modified Moore method Calculus section the more facile it *seems* to be for them to adapt to the MMM and the feed-back is typically quite nice. That is not to say that for a student whose first exposure to the MMM is in Calculus III necessarily does not take to it or do well. In the Fall of 2008, Calculus III that the author taught he had 30 students; 22 completed the course and 8 dropped the course during the semester. All eight who dropped had not been enrolled in a MMM class previously or concurrently. Of the twenty-two completing the course, nine had been enrolled in a MMM class previously and their average grade was approximately 3.04 with a standard deviation of approximately 1.12 and a coefficient of skewness of approximately –0.416. Of the twenty-four completing the course, thirteen had been not enrolled in a MMM class previously and their average grade was approximately 2.92 with a standard deviation of approximately 0.86 and a coefficient of skewness of approximately 0.164. So, the grades did not differ practically but what was of interest was the negative skew for the group that had been enrolled in a MMM class previously versus the positive skew of the group that had not been enrolled in a MMM class previously. The drop rate of 8 of 30 is close to the average drop rate for the author's classes over the last two decades.

Students who have done research with the author or whom the author has directed for Senior Seminar or Honours thesis work within the past decade, twelve had at least one Calculus course with the author and eleven did not (but all twenty-three had at least two courses with the author). Indeed, ten of the twenty-three have gone on to graduate school (and two are planning to go whilst three are still undergraduates). Within the past decade, there were many more students who had a MMM class with the author who did thesis work, did research with a colleague of the author, or went on to graduate school.

We define this record as a success and the success is entirely the students' not the author's; but, getting students to study more mathematics is 'a good thing.'[44]

At KUP, most (if not all) colleagues of the author are decidedly more traditional or reform and there are a number of students who gravitate toward the colleagues and away from the author. It is the position of the author that this is also fine for the educational experience should not be a miserable, hideous, and torturous exercise (though in many instances it is) so minimisation of negative stress and misery is a fine goal. Notwithstanding, there have been some notable failures where the author has failed to inspire students to meet their potential. Several times students have begun class with the author but slipped away due to circumstances such as personal reasons, family trouble, the student worked a night shift, etc. and sometimes due to the author's unwillingness to push a student. There have been students (easily recalled from Fall of 2008 semester) who were almost physically

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[44]The only success for the author with these numbers is that he has a 'knack' for identifying gifted students (who sometimes do not realise that they are gifted and it takes much work to help the realise that they are mathematically gifted).

sick for each class with the author - - two in particular clearly hated the modified Moore method (or the author, perhaps both) - -one could reasonably say this was an example of a failure. Indeed, if the student realised mathematics is not that student's 'cup of tea,' and then later found something that was, then we should not characterise it as a failure; but, for now such does not seem to be the case.

At MC, the author found for the fifteen year period of $1990 - 2004$, most (approximately 0.97) students who took his Calculus course and passed it whilst then taking the subsequent course passed the subsequent course and most (approximately 0.53) students who took his Calculus course and passed it whilst then taking the subsequent course earned a grade equal to or greater than the grade earned in the author's course. No data has been complied at KUP.

VI. Conclusion.

In sum, the author described his attempts to establish and enact inquiry-based learning (IBL) via a modified Moore method (MMM) in the Calculus sequence and how challenging it is to do so, how part of the reason for doing this is to encourage undergraduate to think for themselves, enhance the learning experience undergraduates have when taking Calculus, and how this programme of instruction might lead or direct undergraduates to do mathematical research after the Calculus sequence. We outlined some of the strategies employed; summarised the modified Moore method use; and discussed several types of problems where authentic IBL may indeed take place within the confines of the Calculus sequence. This method employs elements of the classic Moore method (students **doing** rather than seeing, hearing, or reading) which creates a moderate pace for the course; not too fast nor too slow (hopefully), for each of the classes taught within the Calculus sequence.

Perhaps the most important part of MMM the author uses is one should remain flexible, attempt to be moderate in tone and attitude, be willing to adjust dependent upon the conditions of the class, and not be doctrinaire about methods of teaching or assisting student learning. It is my belief that this method maximises educational opportunity for the most students by attempting to teach to as heterogeneous a group as possible and by attempting to incorporate encouragement, understanding, and some of the themes that Nel Noddings calls the 'ethic of caring' [46] whilst not creating an pseudo-mathematics experience for the students or abandoning rigor and precision. For each individual instructor, the method employed should be that which is most comfortable for him and connects with the students.

I opine that this pseudo-Socratic method should be considered by more instructors of mathematics. I deem this because many of the students taught in this method have gone on to graduate school or entered the work-force and have communicated with me that they felt that these courses taught in this manner were the most educationally meaningful for them. The inclusion of open discussion and student presentation component to the course is key so that the experience for the students is neither drill drudgery nor push button calculator nonsense; creating a didactic experience with discussion, debate, and deliberation establishes an atmosphere that encourages thought. One must wonder in order to hypothesise; one must ponder in order to gain insight; and, one must put in hours and hours of work which leads down wrong paths in order to find the correct path.

Whilst a student myself over the course of many years, I was exposed to many different methods of instruction and I found that each had its strengths and weaknesses. I can honestly say that I moderately succeeded in almost every class taught with the Moore or modified Moore method as well as in traditional or reform settings. But, I was inspired and learnt more, I opine, in a Moore or modified Moore setting. *That which I learnt the best was that which I did myself, rather than be told about, lectured to, or even read about.* I must *do* in order to *understand*. That I can not explain something does not mean it does not exist, it simply means that it is unknowable (at this point or perhaps it is never knowable).

This modified Moore method seeks to minimise the amount of lectures; but, allows for students to read from multiple sources and converse (after presentations); it acknowledges that learning is a never-ending process rather than a commodity or entity that can be given like the metaphor of an instructor cracking open the head of a student then pouring the knowledge into said head; and, it is an attempt

to establish authentic and meaningful inquiry-based learning in mathematics. In that regard it is very much reminiscent of reform methods and the philosophy of John Dewey, "the traditional scheme is, in essence, one of imposition from above and from outside."[45] The MMM scheme is one of lateral cooperation with guidance and facilitation from the instructor and hard work, perseverance, and patience from the students.

Obviously, the MMM dialectic shares many elements in common to the Moore method of instructor questioning of students so that the students may forge a solution. However, the MMM method seeks to minimise but not eliminate the use of 'leading' questions being posed by the instructor. The students and instructor control the dialectic, rather than the instructor as in Plato's *Meno*. The guidance from the instructor is in the construction of the material to be covered and the pertinent questions posed that require more than dichotomous responses. This is not the same as in the *Meno* where Socrates leads the slave boy through a series of questions so constructed that they require merely a 'yes' or 'no' response. The queries contain open questions from the perspective of the students (and perhaps the instructor) without indication as to whether they are true or false under the axioms assumed.

I try to include in every paper I write the story of P. J. Halmos recalling a conversation with R. L. Moore where Moore quoted a Chinese proverb; that proverb provides a summation of the justification of the MMM method employed in teaching the Calculus sequence. It states, "I see, I forget; I hear, I remember; I *do*, I *understand*." It is in that spirit that a core point of the argument presented in the paper is that IBL seems to be effective in creating an atmosphere in a course that is educationally meaningful for the students and in which encourages them to attempt to *do* mathematics. In addition, it seems to assist the student to progress from an elementary understanding to a more refined understanding of mathematics. An innovation in the pedagogy proposed is that not all questions posed in the courses are answered and that most of the questions that are posed for students to ponder and attempt are actually posed by the students themselves. Many of the questions put forward in the courses are left for the student to ponder during the student's matriculation and answer at a later date. Examples of processes, problem-solving techniques, solutions, proofs, counterexamples, etc. are given but *most* of the actual work is done by the students.

So, this paper argues for the use of a modified Moore method (MMM) to create an authentic inquiry-based learning (IBL) environment and presents the model for use the MMM in the Calculus sequence. Further, it is proposed that the MMM assists students to establish a firm foundation for subsequent course work.

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[45]John Dewey, *Experience and Education* New York: Macmillan, 1938), page 18.

References

[1] Ball, Deborah; Thames, Mark; &., Phelps, Geoffrey, "Content Knowledge for Teaching: What Makes It Special?" *Journal of Teacher Education, 59, no. 5* (2008): 389 - 407.

[2] Chalice, Donald R., "'How to Teach a Class by the Modified Moore Method." *American Mathematical Monthly, 102*, no. 4 (1995) 317 - 321.

[3] Cohen, David W., "A Modified Moore Method for Teaching Undergraduate Mathematics." *American Mathematical Monthly* 89, no. 7 (1982): 473 - 474, 487 - 490.

[4] Conference Board of the Mathematical Sciences, *Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States,* Washington, DC, Mathematical Association of America and the American Mathematical Society, 1995.

[5] Conference Board of the Mathematical Sciences, *The Mathematical Education of Teachers,* Washington, DC: Mathematical Association of America, 2001.

[6] Committee on the Undergraduate Program in Mathematics, *Pre-graduate Preparation of Research Mathematicians*, Washington, DC, Mathematical Association of America,1963.

[7] Committee on the Undergraduate Program in Mathematics, *A General Curriculum in Mathematics for College*, Washington, DC, Mathematical Association of America,1965.

[8] Committee on the Undergraduate Program in Mathematics, *Reshaping College Mathematics*, Washington, DC, Mathematical Association of America, 1989.

[9] Committee on the Undergraduate Program in Mathematics, *The Undergraduate Major in the Mathematical Sciences*, Washington, DC, Mathematical Association of America,1991.

[10] Committee on the Undergraduate Program in Mathematics, *CUPM Discussion Papers about the Mathematical Sciences in 2010: What Should Students Know?*, Washington, DC, Mathematical Association of America, 2001.

[11] Committee on the Undergraduate Program in Mathematics, *Guidelines for Programs and Departments in Undergraduate Mathematical Sciences (Working Paper)*,Washington, DC, Mathematical Association of America, 2001.

[12] Committee on the Undergraduate Program in Mathematics, *CUPM Interim Reports: Toward a Working Draft Curriculum Guide (Working Paper)*, Washington, DC,Mathematical Association of America, 2001.

[13] Committee on the Undergraduate Program in Mathematics, *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004*, Washington, DC,Mathematical Association of America, 2004.

[14] Davis, Miriam. "Creative Mathematics Instruction: The Method of R. L. Moore," Ph.D. dissertation, Auburn University, Auburn, AL, 1970.

[15] Dewey, John, *Democracy and Education*. New York: Macmillan, 1916.

[16] Dewey, John, *Experience and Education*. New York: Macmillan, 1938.

[17] Dewey, John, *Logic: The Theory of Inquiry*. New York: Holt, 1938.

[18] Duren, Lowell R., "An Adaptation of the Moore Method to the Teaching of Undergraduate Real Analysis - - A Case Study Report." Ph.D. dissertation, The Ohio State University, 1970.

[19] Fitzpatrick, Benjamin, Jr., "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 41 - 45.

[20] Forbes, D. R., "The Texas System: R. L. Moore's Original Edition." Ph.D. dissertation, University of Wisconsin, Madison, WI, 1971.

[21] Foster, James A., Barnett, Michael, & Van Houten, Kare, "(In)formal methods: Teaching Program Derivation via the Moore Method." *Computer Science Education*, 6, no. 1 (1995): 67 - 91.

[22] Frantz, J. B., The Moore Method. In *Forty Acre Follies*, 111 - 122. Dallas, TX:Texas Monthly Press, 1983.

[23] Gersting, Judith L. and Kuczkowski, Joseph E., "Why and How to Use Small Groups in the Mathematics Classroom." *College Mathematics Journal* 8, no. 2 (1977): 270 - 274.

[24] Gold B., Marion W. & S. Keith. *Assessment Practices in Undergraduate Mathematics*. MAA Notes, 49. Washington, DC: Mathematical Association of America, 1999.

[25] Guba, Egon and Lincoln, Yvonna. *Naturalistic Inquiry*, Newbury Park, CA: Sage, 1985.

[26] Halmos, Paul R., How To Teach. In *I Want To Be A Mathematician.* New York:Springer-Verlag, 1985.

[27] Halmos, Paul R., *Naïve Set Theory,* Princeton, NJ: D. Van Nostrand, 1960.

[28] Halmos, Paul R., "What Is Teaching?" *American Mathematical Monthly* 101, no. 9 (1994): 848 - 854.

[29] Halmos, Paul R., Moise, Edwin E. and Piranian, George, "The Problem of Learning to Teach." *American Mathematical Monthly* 82, no. 5 (1975) 466 - 476.

[30] Halonen, J. S. , Brown-Anderson, F., McKeachie, W. J. "Teaching Thinking." In *McKeachie's Teaching Tips: Strategies, Research, and Theory for College and University Teachers (11th ed.)*, Boston: Houghton-Mifflin, 2002.

[31] Hodkinson, Phillip, "Learning as Cultural and Relational: Moving Past Some Troubling Dualisms," *Cambridge Journal of Education, 35, no. 1* (2005): 107 - 119.

[32] Jones, F. B. "The Moore Method." *American Mathematical Monthly* 84, no. 4 (1977): 273 - 278.

[33] Kirk, Jerome & Miller, Marc. *Reliability and Validity in Qualitative Research*, Newbury Park, CA: Sage, 1986.

[34] McLoughlin, M. P. M. M. "The Central Role of Proof in the Mathematics Canon: The efficacy of teaching students to create proofs using a fusion of modified Moore, traditional, and reform methods." Paper presented at the annual summer meeting of the Mathematical Association of America, Burlington, Vermont, 2002.

[35] McLoughlin, M. P. M. M. "Initiating and Continuing Undergraduate Research in Mathematics: Using the fusion method of traditional, Moore, and constructivism to encourage, enhance, and establish undergraduate research in mathematics." Paper presented at the annual meeting of the Mathematical Association of America, Baltimore, Maryland, 2003.

[36] McLoughlin, M. P. M. M. 'The Fusion Method of Traditional, Moore, and Constructivism and the Incorporation of Undergraduate Research Throughout the Mathematics Curriculum." Paper presented at the annual meeting of the Mathematical Association of America, Baltimore, Maryland, 2003.

[37] McLoughlin, M. P. M. M. "On the Nature of Mathematical Thought and Inquiry: A Prelusive Suggestion." Paper presented at the annual meeting of the Mathematical Association of America, Phoenix, Arizona, 2004.

[38] McLoughlin, M. P. M. M. "Crossing the Bridge to Higher Mathematics: Using a modified Moore approach to assist students transitioning to higher mathematics." Paper presented at the annual meeting of the Mathematical Association of America, San Diego, California, 2008.

[39] McLoughlin, M. P. M. M. "Inquiry Based Learning: A Modified Moore Method Approach To Encourage Student Research." Paper presented at the Legacy of R. L. Moore Conference, Austin, TX, 2008.

[40] McLoughlin, M. P. M. M. "Inquiry Based Learning: A Modified Moore Method Approach To Encourage Student Research." Paper presented at the annual meeting of the American Statistical Association, Denver, CO, 2008.

[41] McLoughlin, M. P. M. M. " Inquiry-Based Learning: An Educational Reform Based Upon Content-Centred Teaching." Paper presented at the annual meeting of the American Mathematical Society, Washington, DC, 2009.

[42] Moise, Edwin E. "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, no. 4 (1965): 407 - 412.

[43] Murtha, J. A. "The Apprentice System for Math Methods." *American Mathematical Monthly* 84, no. 6 (1977): 473 - 476.

[44] National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics* (1989). Reston, VA: NCTM.

[45] National Science Board Committee on Undergraduate Science and Engineering Education, *Undergraduate Science, Mathematics, and Engineering Education*, Washington, DC, National Science Board, 1986.

[46] Noddings, Nel, *Philosophy of Education.* Boulder, CO: Westview, 1995.

[47] O'Shea, D. & H. Pollatsek, "Do We Need Prerequisites?" *Notices of the American Mathematical Society,* 44, no. 5 (1997): 564 – 570.

[48] Page, Warren "A Small Group Strategy for Enhancing Learning." *American Mathematical Monthly* 86, no. 9 (1979): 856 - 858.

[49] Pólya, George *How to Solve It.* New York: Doubleday, 1957.

[50] Rasmussen, Chris and Marrongelle, Karen, "Pedagogical Content Tools: Integrating Student Reasoning and Mathematics in Instruction," *Journal for Research in Mathematics Education, 37, no. n5* (2006): 388 - 420.

[51] Sax, Gilbert, *Principles of Educational and Psychological Measurement and Evaluation* 3rd Ed. Belmont, CA: Wadsworth, 1989.

[52] Siebert, D. Draper, R. J., "Why Content-Area Literacy Messages Do Not Speak to Mathematics Teachers: A Critical Content Analysis," *Literacy Research and Instruction, 47, no. 4* (2008): 229 - 245.

[53] Solow, Daniel, *How to Read and Do Proofs* 3rd Ed. New York: Wiley, 2002.

[54] Steffe, Leslie and Kieren, Thomas, "Radical Constructivism and Mathematics Education," *Journal for Research in Mathematics Education* 25, no. 6 (1994): 711 – 733.

[55] Stinton, D., Bidwell, C., Jett, C., Powell, G., and Thurman, M., "Critical Mathematics Pedagogy: Transforming Teachers' Practices," 2008 (http://www.gsu.edu/stinton).

[56] Swales, Christine. *Editing Distance Education Materials. Knowledge Series.* http://www.col.org/Knowledge/KSediting.pdf (2000)

[57] Tucker, Adam, *Models That Work: Case Studies in Effective Undergraduate Mathematics Programs.* MAA Notes, 38. Washington, DC: Mathematical Association of America, 1995.

[58] Velleman, Daniel, *How to Prove It: A Structured Approach* New York: Cambridge University Press, 1994.

[59] Weissglagg, Julian "Small Groups: An Alternative to the Lecture Method." *College Mathematics Journal* 7, no. 1 (1976): 15 - 20.

[60] Wilder, R. L. "The Nature of Mathematical Proof." *American Mathematical Monthly* 51, no. 6 (1944): 309 - 325.

[61] Wilder, R. L. "The Role of The Axiomatic Method." *American Mathematical Monthly* 74, no. 2 (1967): 115 - 127.

[62] Wilder, R. L. Axiomatics and the Development of Creative Talents. In *The Axiomatic Method with Special Reference to Geometry and Physics*, edited by L.Henkin, P. Suppes, and A. Tarski. Amsterdam: North - Holland, 1976.

[63] Wilder, R. L. "Robert Lee Moore, 1882 - 1974." *Bulletin of the American Mathematical Society* 82, no. 3 (1976): 417 - 427.

[64] Wilder, R. L. The Mathematical Work of R. L. Moore: Its Background, Nature, and Influence. In *A Century of Mathematics in America: Part III*, edited by Peter Duren. Providence, RI: American Mathematical Society, 1989.

[65] Whyburn, Lucille S. "Student Oriented Teaching - The Moore Method." *American Mathematical Monthly* 77, no. 4 (1970): 351 - 359.

[66] Yoder, Janice D. and Hochevar, Catherine M., "Encouraging Active Learning Can Improve Students' Performance on Examinations," *Teaching of Psychology, 32, no. 2* (2005): 91 - 95.